a horizontal layer heated from below had been previously investigated [7]. The transverse motion in a horizontal layer also leads to increased stability. The critical Rayleigh numbers increase monotonically with increasing Peclet number; the closing of levels is in this case absent. There is thus a similarity with spatial perturbations in a vertical layer. However, when comparing the results of [7] with those derived here, it should be stressed that there is no complete analogy between the two problems. In the case of the horizontal layer the transverse motion is directed across the unperturbed isotherms resulting in the decrease of the unstably stratified layer thickness with increasing velocity of the transverse motion. The transverse motion in a vertical layer occurs, on the other hand, along the isotherms without distorting the temperature distribution equilibrium.

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A HYDRODYNAMIC MODEL OF DISPERSE SYSTEMS

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The dynamic equations of motion of the phases of a monodisperse system are formulated in the approximation of interpenetrating interacting continua. The energy transfer equations of the pulsations of the phases in various directions are derived. These serve to close the above system of dynamic equations.

We investigated the non-Newtonian hydromechanics of disperse systems in [1] and extended it to gas suspensions in [2, 3]. The approach used in [1–3] is to some extent phenomenological, in that the random pulsations of the phases of a disperse system are dealt with on the basis of the equations of motion of the phases postulated *a priori* as for continua, whereas strictly speaking such equations can only be posited without contradiction after such analysis. We now propose to eliminate the contradiction by deriving the equations of phase motion in the continuum approximation in a very natural way.

1. The model and the dynamic equations. Let us consider a system of particles of radius a and density d_2 suspended in a medium of density d_1 and viscosity μ_0. We shall specify the state of this system by means of the ensemble-average values of the volume concentration $\langle\rho\rangle$, the velocity of the dispersed phase $\langle\mathbf{w}\rangle$, the velocity of the fluid medium $\langle\mathbf{v}\rangle$, and the pressure in the medium $\langle p\rangle$. Fulfilment of the ergodic hypothesis (which we assume) means that these average values, referred to from now on as the "dynamic variables", coincide with the quantities obtained by averaging the indicated quantities over volume of a mixture containing $N \gg 1$ particles and then taking the limit as $N \to \infty$. The instantaneous local values of the velocity $\mathbf{w}$ of a single particle, of the average velocity $\mathbf{v}$ of the fluid in its specific volume σ, of the average pressure p in this volume, and of the quantity $\rho = \sigma_0 / \sigma$, where $\sigma_0 = {}^4/_3\pi a^3$ (which is the local value of the volume concentration of the disperse system) differ from the above average values by certain random addends denoted by primes. These random quantities represent the pulsations of the system phases, which we refer to from now on as "quasi-turbulent" pulsations. The specific volume of such a particle can therefore be taken (as in [1–3]) as the smallest "cell" for which determination of the local values of the parameters characterizing the fluid phase is still meaningful. The equation of motion of a single particle can be written as

$$m\frac{d\mathbf{w}}{dt} = m\mathbf{g} + \mathbf{F} + \mathbf{F}_i, \quad \mathbf{F} = \mathbf{F}_p + \mathbf{F}_\mu + \mathbf{F}_\xi + \mathbf{F}_B, \quad m = d_2\sigma_0 \qquad (1.1)$$

Here $\mathbf{g}$ is the acceleration of the external mass field, $\mathbf{F}_i$ is the random force associated with direct interactions ("collisions") of an isolated particle with its neighbors, and $\mathbf{F}$ is the force of interaction of a particle with the supporting fluid stream. The latter force consists of the following components: the force $\mathbf{F}_p$ due to the pressure gradient in the fluid, the force $\mathbf{F}_\mu$ of viscous interaction with the stream, the force $\mathbf{F}_\xi$ due to the excess inertia of the fluid during accelerated relative motion of a particle (i.e. to the additional mass effects), and the Basset force $\mathbf{F}_B$. These forces are given by the expressions

$$\mathbf{F}_p = -\sigma_0\frac{\partial p}{\partial \mathbf{r}}, \quad \mathbf{F}_\mu = d_1\sigma_0\beta K(\rho)\,\mathbf{u}, \quad \mathbf{F}_\xi = d_1\xi\sigma_0\frac{d\mathbf{u}}{dt}, \quad \mathbf{u} = \mathbf{v} - \mathbf{w}$$
$$\mathbf{F}_B = d_1\sigma_0\gamma\int_{-\infty}^{t}\gamma'\frac{d\mathbf{u}}{dt'}\frac{dt'}{\sqrt{t-t'}}, \quad \beta = \frac{9\nu_0}{2a^2}, \quad \gamma = \frac{9}{2a}\left(\frac{\nu_0}{\pi}\right)^{1/2}, \quad \nu_0 = \frac{\mu_0}{d_1} \qquad (1.2)$$

where ρ, p and $\mathbf{v}$ are defined, as noted above, within the specific volume of a given particle, ξ is the additional mass factor, γ' is a coefficient of the order of unity (for simplicity we assume that ξ and γ' can depend on $\langle\rho\rangle$ but not on ρ; as $\langle\rho\rangle \to 0$ we have $\xi = {}^1/_2$, $\gamma' = 1$), and $K(\rho)$ is a function which allows for the deviation of the force $\mathbf{F}_\mu$ from the Stokes force for constrained flow past the particle (the quantity $K(\rho)$ can, of course, depend on the dynamic variables).

Let us introduce the particle velocity distribution function $f(\mathbf{w}; \mathbf{r}, t)$ normed to the average countable particle concentration $n(\mathbf{r}, t)$; let us also introduce the nominal distribution functions $f(\mathbf{v}; \mathbf{r}, t \mid \mathbf{w})$, $f(\rho; \mathbf{r}, t \mid \mathbf{v}, \mathbf{w})$ and $f(p; \mathbf{r}, t \mid \rho, \mathbf{v}, \mathbf{w})$ normed to unity. The quantity $f(\mathbf{v}; \mathbf{r}, t \mid \mathbf{w})\,d\mathbf{v}$, for example, represents the probability with which the velocity of the liquid phase in the specific volume of a particle moving at the velocity $\mathbf{w}$ will lie in the range $(\mathbf{v}, \mathbf{v} + d\mathbf{v})$. The operation of averaging over the ensemble is defined as follows:

$$\langle\varphi\rangle = \frac{1}{n(\mathbf{r},t)}\int \varphi f(p;\mathbf{r},t\,|\,\rho,\mathbf{v},\mathbf{w})\,f(\rho;\mathbf{r},t\,|\,\mathbf{v},\mathbf{w})\,f(\mathbf{v};\mathbf{r},t\,|\,\mathbf{w})\,f(\mathbf{w};\mathbf{r},t)\,dA$$

$$dA = dp\,d\rho\,d\mathbf{v}\,d\mathbf{w}$$

Clearly, we can assume in the general case that the above nominal distribution functions constitute certain functionals of the unary function $f(\mathbf{w};\mathbf{r},t)$ and of the subsequent multicomponent distribution functions. The form of these functionals is not known. Since the force $\mathbf{F}$ of (1.1) depends on p, ρ and $\mathbf{v}$, the kinetic equation for $f(\mathbf{w};\mathbf{r},t)$ must also contain the above multicomponent distribution functions, although it is impossible to write out a chain of kinetic equations for all such functions (*). However, if we average over the nominal distributions directly, then we can use the standard procedure to obtain the kinetic equation for $f(\mathbf{w};\mathbf{r},t)$ from the Liouville and Hamilton equations. The resulting kinetic equation is of the form

$$\frac{Df}{Dt} + \mathbf{w}'\frac{\partial f}{\partial \mathbf{r}} + \left(\mathbf{F}^* - \frac{D\langle\mathbf{w}\rangle}{Dt}\right)\frac{\partial f}{\partial \mathbf{w}'} - \left(\frac{\partial f}{\partial \mathbf{w}'} * \mathbf{w}'\right) : \left(\frac{\partial}{\partial \mathbf{r}} * \langle\mathbf{w}\rangle\right) = \left(\frac{\partial f}{\partial t}\right)_i$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \langle\mathbf{w}\rangle\frac{\partial}{\partial \mathbf{r}}, \quad \mathbf{a}*\mathbf{b} = \|a_i b_j\|, \quad \mathbf{A}:\mathbf{B} = A_{ij}B_{ji} \tag{1.3}$$

$$\mathbf{F}^* = \int \mathbf{F} f(p;\mathbf{r},t\,|\,\rho,\mathbf{v},\mathbf{w})\,f(\rho;\mathbf{r},t\,|\,\mathbf{v},\mathbf{w})\,f(\mathbf{v};\mathbf{r},t\,|\,\mathbf{w})\,dp\,d\rho\,d\mathbf{v}$$

The term in the right side of (1.3) describes the variation of $f(\mathbf{w};\mathbf{r},t)$ as a result of collisions between particles. The number of particles in the system does not depend on the collisions. This leads us to assume that the total momentum and the total energy of the colliding particles are invariant under collisions (the meaning of this assumption is self-evident). We then use the standard procedure [5] to obtain from (1.3) the equations of conservation of the mass and momentum of the dispersed phase in the continuum approximation, namely

$$\frac{D\langle\rho\rangle}{Dt} + \langle\rho\rangle\frac{\partial\langle\mathbf{w}\rangle}{\partial\mathbf{r}} = 0, \quad \langle\rho\rangle \equiv n\sigma_0$$

$$d_2\langle\rho\rangle\frac{D\langle\mathbf{w}\rangle}{Dt} = -\frac{\partial\mathbf{P}^{(p)}}{\partial\mathbf{r}} + d_2\langle\rho\rangle\mathbf{g} + \langle\rho\rangle(\mathbf{G}_p + \mathbf{G}_\mu + \mathbf{G}_\xi + \mathbf{G}_B) \tag{1.4}$$

$$\mathbf{P}^{(p)} = d_2\langle\rho\rangle\langle\mathbf{w}'*\mathbf{w}'\rangle, \quad \sigma_0\{\mathbf{G}_p, \mathbf{G}_\mu, \mathbf{G}_\xi, \mathbf{G}_B\} = \{\langle\mathbf{F}_p, \mathbf{F}_\mu, \mathbf{F}_\xi, \mathbf{F}_B\rangle\}$$

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*) An attempt to deal with these difficulties in such a way as to break off the chain of kinetic equations at the equation for the unary distribution function was made in [4], where the presence of the force $\mathbf{F}$ was allowed for by introducing a term describing diffusion in the velocity space into the equation. This was essentially equivalent to assuming the random force $\mathbf{F}' = \mathbf{F} - \langle\mathbf{F}\rangle$ to be Markovian, as in the theory of Brownian motion. Such an assumption is invalid in our problem, since the characteristic time T of the variation of $\mathbf{F}'$ coincides with the time of significant alteration of the characteristics of the quasiturbulent motion. If, despite this, we decide to assume a Markovian $\mathbf{F}'$ as an approximation, i.e. if we limit ourselves to the consideration of processes whose duration is at all events larger than T, then the term $(df/dt)_i$ in the kinetic equation cannot, in principle, be written in standard Boltzmann form as in [4], since the characteristic time of system relaxation due to collisions between particles is of the same order as the time of travel τ of a particle between successive collisions, and since it is usually the case that $\tau \ll T$.

A similar equation can also be readily derived for the total pulsation energy of the particles; this equation is also of standard form [5]. However, the quasi-turbulence of disperse systems is usually essentially anisotropic, which focuses interest on the energies of the quasi-turbulent motions in various directions. The forms of the equations for these quantities is considered in more detail in Sect. 4 of the present paper.

The second equation of (1.4) can also be obtained from equation of particle motion (1.1). In fact, let us formally set $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{w}\frac{\partial}{\partial \mathbf{r}}$ then set $\rho = \langle\rho\rangle + \rho'$ in (1.1), etc., and finally sum the resulting equations for $N \gg 1$ particles present in the volume V. This yields

$$Vd_2\langle\rho\rangle\frac{D(\mathbf{w})}{Dt} + m\left[\left(\sum_{k=1}^{N}\mathbf{w}'^{(k)}\frac{\partial\mathbf{w}'^{(k)}}{\partial\mathbf{r}}\right)\langle\mathbf{w}\rangle + \frac{D}{Dt}\sum_{k=1}^{N}\mathbf{w}'^{(k)}\right] +$$

$$+ m\frac{\partial}{\partial\mathbf{r}}\sum_{k=1}^{N}\mathbf{w}'^{(k)}*\mathbf{w}'^{(k)} - m\sum_{k=1}^{N}\mathbf{w}'^{(k)}\frac{\partial\mathbf{w}'^{(k)}}{\partial\mathbf{r}} = V\langle\rho\rangle d_2\mathbf{g} +$$

$$+ V\langle\rho\rangle\mathbf{G} + \sum_{k=1}^{N}(\mathbf{F}^{(k)} - \sigma_0\mathbf{G} + \mathbf{F}_i^{(k)}), \quad \sigma_0\mathbf{G} = \langle\mathbf{F}\rangle$$

As $N \to \infty$ the second term in the left side and the last term in the right side of this equation clearly vanish, so that we can divide this equation by V and take the limit as $N \to \infty$ to obtain

$$d_2\langle\rho\rangle\frac{D\langle\mathbf{w}\rangle}{Dt} = -\frac{\partial\mathbf{P}^{(p)}}{\partial\mathbf{r}} + m\left\langle\mathbf{w}'\frac{\partial\mathbf{w}'}{\partial\mathbf{r}}\right\rangle + d_2\langle\rho\rangle\mathbf{g} + \langle\rho\rangle\mathbf{G}$$

Comparing this equation with (1.4), we find that it is necessarily the case that $\langle\mathbf{w}'(\nabla\mathbf{w}')\rangle = 0$. We also assume that $\langle\mathbf{v}'(\nabla\mathbf{w}')\rangle = 0$ in our computation of $\mathbf{G}_\xi$ below.

In order to particularize the second equation of (1.4) we must find explicit expressions for the quantities $\mathbf{G}_p$, $\mathbf{G}_\mu$, $\mathbf{G}_\xi$ and $\mathbf{G}_B$. Making use of relations (1.2), we obtain the following expressions accurate to within second-order terms in the quasi-turbulent quantities:

$$\mathbf{G}_p = -\frac{\partial\langle p\rangle}{\partial\mathbf{r}}, \quad \mathbf{F}_p' = -\sigma_0\frac{\partial p'}{\partial\mathbf{r}}$$

$$\mathbf{G}_\mu = \beta d_1\left(K\langle\mathbf{u}\rangle + \frac{dK}{d\langle\rho\rangle}\langle\rho'\mathbf{u}'\rangle + \frac{1}{2}\frac{d^2K}{d\langle\rho\rangle^2}\langle\mathbf{u}\rangle\langle\rho'^2\rangle\right)$$

$$\mathbf{F}_\mu' = \sigma_0\beta d_1\left(K\mathbf{u}' + \frac{dK}{d\langle\rho\rangle}\langle\mathbf{u}\rangle\rho'\right), \quad K = K(\langle\rho\rangle)$$

$$\mathbf{G}_\xi = \xi d_1\left(\frac{D\langle\mathbf{u}\rangle}{Dt} + \frac{\partial\mathbf{P}_\xi}{d_1\partial\mathbf{r}}\right), \quad \mathbf{F}_\xi' = \xi d_1\sigma_0\left(\frac{D\mathbf{u}'}{Dt} + \left(\mathbf{w}'\frac{\partial}{\partial\mathbf{r}}\right)\langle\mathbf{u}\rangle\right)$$

$$\mathbf{G}_B = \gamma d_1\int_{-\infty}^{t}\gamma'\left(\frac{D\langle\mathbf{u}\rangle}{Dt} + \frac{\partial\mathbf{P}_\xi}{d_1\partial\mathbf{r}}\right)\Big|_{t=t'}\frac{dt'}{\sqrt{t-t'}}$$

$$\mathbf{F}_B' = \gamma d_1\sigma_0\int_{-\infty}^{t}\gamma'\left(\frac{D\mathbf{u}'}{Dt} + \left(\mathbf{w}'\frac{\partial}{\partial\mathbf{r}}\right)\langle\mathbf{u}\rangle\right)\Big|_{t=t'}\frac{dt'}{\sqrt{t-t'}} \qquad (1.5)$$

$$\mathbf{P}_\xi = d_1\langle\mathbf{w}'*\mathbf{u}'\rangle$$

For convenience, we shall henceforth take the average specific volume of a single particle in the system $\langle\sigma\rangle = \sigma_0\langle\rho\rangle^{-1}$ as our unit volume. The equations of conservation of the mass and momentum of the fluid phase can be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\rho - (1-\rho)\frac{\partial \mathbf{v}}{\partial \mathbf{r}} = 0, \quad \mathbf{e} = \left\| \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right\|$$
$$d_1(1-\rho)\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{v} = -\frac{\partial p}{\partial \mathbf{r}} + \frac{\partial(\mu\mathbf{e})}{\partial \mathbf{r}} + d_1(1-\rho)\mathbf{g} - \mathbf{F}, \quad \mu = \mu_0 S \tag{1.6}$$

Here μ is the effective viscosity of the fluid flowing through the lattice relative to the stationary particles. We assume from now on that μ can depend on $\langle\rho\rangle$ but not on ρ. Substituting $\rho = \langle\rho\rangle + \rho'$ into (1.6), etc., and then averaging, we obtain the following equations (accurate to within second-order terms) describing the average motion of the fluid in the disperse system:

$$\left(\frac{\partial}{\partial t} + \langle\mathbf{v}\rangle\frac{\partial}{\partial \mathbf{r}}\right)\langle\rho\rangle - (1-\langle\rho\rangle)\frac{\partial\langle\mathbf{v}\rangle}{\partial \mathbf{r}} - \frac{\partial \mathbf{q}}{\partial \mathbf{r}} = 0, \quad \mathbf{q} = -\langle\rho'\mathbf{v}'\rangle$$
$$d_1\left[\frac{\partial}{\partial t}((1-\langle\rho\rangle)\langle\mathbf{v}\rangle) + \frac{\partial}{\partial \mathbf{r}}((1-\langle\rho\rangle)\langle\mathbf{v}\rangle * \langle\mathbf{v}\rangle)\right] + d_1\frac{\partial \mathbf{q}}{\partial t} =$$
$$= -\frac{\partial \mathbf{P}^{(f)}}{\partial \mathbf{r}} - \frac{\partial\langle p\rangle}{\partial \mathbf{r}} + \frac{\partial(\mu\langle\mathbf{e}\rangle)}{\partial \mathbf{r}} + d_1(1-\langle\rho\rangle)\mathbf{g} - \mathbf{G}$$
$$\mathbf{P}^{(f)} = d_1[(1-\langle\rho\rangle)\langle\mathbf{v}' * \mathbf{v}'\rangle + \mathbf{q} * \langle\mathbf{v}\rangle + \langle\mathbf{v}\rangle * \mathbf{q}] \tag{1.7}$$

The force G appearing in this expression is the sum of the forces $\mathbf{G}_p$, $\mathbf{G}_\mu$, $\mathbf{G}_\xi$ and $\mathbf{G}_B$ of (1.5). Together with (1.4), Eqs. (1.7) constitute the system of dynamic equations describing the average motion of the disperse system. These equations differ markedly from the equations of average motion usually postulated in the phenomenological theory of multicomponent media (e.g. see [6, 7]. In the first place, an essentially new term appears in the equation of conservation of the mass of the fluid phase. This term is associated with the three-dimensional quasi-turbulent fluid flux **q**. The same situation occurs in formulating the continuity equation for the average motion of an agitated compressible fluid. The equations of conservation of the momenta of the phases contain new terms describing the additional variation of the momentum occasioned by quasi-turbulence. These terms are due (a) to the pulsation pressure $\mathbf{P}^{(f)}$ and $\mathbf{P}^{(p)}$, and (b) to local variation of the fluid momentum transferred by the flux **q**.

The second equations of (1.4) and (1.7) are approximate. This is due to the fact that we omitted terms of higher than the second order in the primed quantities in computing the force **G** of (1.5) and in transforming the left side of the second equation of (1.6). If the quasi-turbulent quantities are relatively large, then this can result in substantial errors in the dynamic quantities being determined. However, this usually applies to particle suspensions when the momentum of the gas, and therefore the forces $\mathbf{G}_\xi$ and $\mathbf{G}_B$, are entirely negligible. The indicated error is then due solely to the form of $\mathbf{G}_\mu$ in (1.5). It appears to be convenient in certain cases to replace the $\mathbf{G}_\mu$ in (1.4) and (1.7) by one of the empirical relations for the viscous interaction force, i.e. by a relation of the form

$$\mathbf{G}_\mu \approx \beta d_1 K^*(\langle\rho\rangle)\langle\mathbf{u}\rangle$$

Many formulas of this type have been accumulated through quasi-fluidization experiments. The function $K^*(\langle\rho\rangle)$ in the above expression describes the viscous resistance of the particle layer with allowance for the particle pulsations; the function $K(\langle\rho\rangle)$ used above is the viscous resistance of the layer relative to the stationary "fixed" particles. The difference between these two resistances is emphasized in most studies on the hydraulics of quasi-fluidized layers. It is considered theoretically in [8], where the second correction term in the expression for G_μ in (1.5) is allowed for. However, the author of [8] uses this term to explain the smaller value of the effective resistance of the quasi-fluidized layer as compared with the resistance of a stationary column packing of the same porosity; we see from (1.5), however, that when $d^2K / d\langle\rho\rangle^2 > 0$ as is the case in all experiments, the indicated term represents the relative increase in the effective hydraulic resistance.

The pulsations of the dynamic variables in liquid suspensions are usually so small that Eqs. (1.4) and (1.7) are adequate approximations of reality.

Complete determination of the above dynamic equations clearly requires us to find a method for calculating the mean-square values of the indicated quasi-turbulent variables. Such a method is described below.

2. Stochastic equations and expressions for the random processes. Using the results and conclusions of [1–3], we can make the following two assumptions about the character of the quasi-turbulence under investigation:

1) We assume that the time and space scales of variation of the dynamic variables are much larger than the corresponding scales of variation of the pulsation quantities associated with the quasi-turbulent motion.

2) The time scale τ of significant change of the quantities of the type $\varphi' = a'b' - \langle a'b'\rangle$, where a' and b' denote any quasi-turbulent variables, is much smaller than the time scale T of variation in the correlation functions $\langle a'b'\rangle$. Therefore in analyzing processes whose characteristic time is of the order of t or higher (where $t \gg \tau$ but $t \ll T$), we can assume that the quantities φ' are Markovian. By analogy with [9], we call the parameters τ and T the "internal" and "external" quasi-turbulence time scales (see also the discussion in [1–3]).

Using these assumptions, subtracting averaged equations (1.1) and (1.6) (i.e. the second equation of (1.4) and Eqs. (1.7)) from the corresponding unaveraged equations (1.1) and (1.6) and averaging the results over the interval $t \gg \tau$, we obtain the following stochastic equations for the random processes ρ', p', $\mathbf{v}'$ and $\mathbf{w}'$ under consideration,

$$m\frac{\partial \mathbf{w}}{\partial t} = \mathbf{F}', \quad \left(\frac{\partial}{\partial t} + \langle \mathbf{u}\rangle \frac{\partial}{\partial \mathbf{r}}\right)\rho' - (1 - \langle\rho\rangle)\frac{\partial \mathbf{v}'}{\partial \mathbf{r}} = 0$$
$$d_1(1 - \langle\rho\rangle)\left(\frac{\partial}{\partial t} + \langle \mathbf{u}\rangle\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{v}' = -\frac{\partial p'}{\partial \mathbf{r}} + \mu\frac{\partial \mathbf{e}'}{\partial \mathbf{r}} - d_1 g\rho' - \mathbf{F}' \qquad (2.1)$$

These equations are written out in the local coordinate system in which the mean velocity of the dispersed phase $\langle \mathbf{w}\rangle$ is equal to zero; in deriving (2.1) we assumed that the averaging of $\mathbf{F}_i$ over $t \gg \tau$ yields zero.

Substituting into (2.1) the relations for $\mathbf{F}'$ from (1.5), we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\frac{\partial}{\partial \mathbf{r}}\right)\rho' - (1-\rho)\frac{\partial \mathbf{v}'}{\partial \mathbf{r}} = 0, \quad \varkappa = \frac{d_1}{d_2}, \quad \pi' = \frac{p'}{d_2}, \quad \nu = \frac{\mu}{d_1}$$
$$\frac{\partial \mathbf{w}'}{\partial t} - \varkappa\xi\frac{\partial \mathbf{u}'}{\partial t} - \varkappa\gamma\Upsilon'\int_{-\infty}^{t}\frac{\partial \mathbf{u}'}{\partial t'}\frac{dt'}{\sqrt{t-t'}} = -\frac{\partial \pi'}{\partial \mathbf{r}} + \varkappa\beta\left(K\mathbf{u}' + \frac{dK}{d\rho}\mathbf{u}\rho'\right)$$
$$\varkappa(1-\rho)\left(\frac{\partial}{\partial t} + \mathbf{u}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{v}' + \varkappa\xi\rho\frac{\partial \mathbf{u}'}{\partial t} + \varkappa\gamma\Upsilon'\rho\int_{-\infty}^{t}\frac{\partial \mathbf{u}'}{\partial t'}\frac{dt'}{\sqrt{t-t'}} =$$
$$= -(1-\rho)\frac{\partial \pi'}{\partial \mathbf{r}} + \varkappa\nu\frac{\partial \mathbf{e}'}{\partial \mathbf{r}} - \varkappa\beta\rho\left(K\mathbf{u}' + \frac{dK}{d\rho}\mathbf{u}\rho'\right) - \varkappa g\rho' \qquad (2.2)$$

Here and in the following we neglect, for simplicity of notation, the angle brackets around the dynamic variables.

Let us represent the random processes in terms of Fourier-Stieltjes integrals. Then we obtain the following equations for the spectral measures dZ_ρ, dZ_π, $d\mathbf{Z}_v$ and $d\mathbf{Z}_w$ of the random processes ρ', π', $\mathbf{v}'$ and $\mathbf{w}'$:

$$(\omega + \mathbf{uk})\, dZ_\rho - (1-\rho)\, \mathbf{k}\, d\mathbf{Z}_v = 0, \quad A(\omega, \mathbf{k})\, d\mathbf{Z}_w - C(\omega, \mathbf{k})\, d\mathbf{Z}_v =$$
$$= -\mathbf{k}\, dZ_\pi + \varkappa\beta \frac{dK}{d\rho} \mathbf{u}\, dZ_\rho, \quad -A_1(\omega, \mathbf{k})\, d\mathbf{Z}_w + C_1(\omega, \mathbf{k})\, d\mathbf{Z}_v =$$
$$= -(1-\rho)\, \mathbf{k}\, dZ_\pi - \varkappa\nu\mathbf{k}(\mathbf{k}\, d\mathbf{Z}_v) - \varkappa\left(\mathbf{g} + \rho\beta \frac{dK}{d\rho} \mathbf{u}\right) dZ_\rho \tag{2.3}$$
$$A = i(1 + \varkappa\xi)\,\omega + (1 + i \operatorname{sign} \omega)\, \varkappa\gamma\gamma' |\omega|^{1/2} + \varkappa\beta K$$
$$A_1 = \varkappa\rho\, [i\xi\omega + (1 + i \operatorname{sign} \omega)\, \gamma\gamma' |\omega|^{1/2} + \beta K]$$
$$C = \varkappa\, [i\xi\omega + (1 + i \operatorname{sign} \omega)\, \gamma\gamma' |\omega|^{1/2} + \beta K]$$
$$C_1 = \varkappa\, [i(1-\rho)(\omega + \mathbf{uk}) + i\xi\rho\omega + (1 + i \operatorname{sign}\omega)\, \gamma\gamma'\rho |\omega|^{1/2} +$$
$$+ \rho\beta K + \nu k^2]$$

Scalar multiplication of the last two equations of (2.3) by the wave vector $\mathbf{k}$ followed by the use of the first equation of (2.3) yields

$$k^2\, dZ_\pi + A(\mathbf{k}\, d\mathbf{Z}_w) = \left(C \frac{\omega + \mathbf{uk}}{1-\rho} + \varkappa\beta \frac{dK}{d\rho} \mathbf{uk}\right) dZ_\rho$$
$$k^2\, dZ_\pi + B(\mathbf{k}\, d\mathbf{Z}_w) = -\left(D \frac{\omega + \mathbf{uk}}{1-\rho} + \varkappa\mathbf{gk}\right) dZ_\rho \tag{2.4}$$
$$B = i\,\omega\rho, \quad D = \varkappa\, [i(1-\rho)(\omega + \mathbf{uk}) + 2\nu k^2]$$

The solution of this system is of the form

$$\mathbf{k}\, d\mathbf{Z}_w = \frac{1}{A-B}\left[(C + D)\frac{\omega + \mathbf{uk}}{1-\rho} + \varkappa\left(\mathbf{g} + \beta \frac{dK}{d\rho} \mathbf{u}\right) \mathbf{k}\right] dZ_\rho$$
$$k^2\, dZ_\pi = \frac{-1}{A-B}\left[(AD + BC)\frac{\omega + \mathbf{uk}}{1-\rho} + \varkappa\left(A\mathbf{g} + B\beta \frac{dK}{d\rho} \mathbf{u}\right) \mathbf{k}\right] dZ_\rho$$

and the solution of (2.3) can be written as

$$d\mathbf{Z}_v = \frac{1}{A_1C - AC_1}\left\{[A_1 + (1-\rho)A]\, \mathbf{k}\, dZ_\pi + \varkappa\left[A\left(\frac{\nu\mathbf{k}(\omega + \mathbf{uk})}{1-\rho} + \right.\right.\right.$$
$$\left.\left.\left. + \mathbf{g} + \rho\beta \frac{dK}{d\rho} \mathbf{u}\right) - A_1\beta \frac{dK}{d\rho} \mathbf{u}\right] dZ_\rho\right\}$$
$$d\mathbf{Z}_w = \frac{1}{A_1C - AC_1}\left\{[C_1 + (1-\rho)C]\, \mathbf{k}\, dZ_\pi + \right.$$
$$\left. + \varkappa\left[C\left(\frac{\nu\mathbf{k}(\omega + \mathbf{uk})}{1-\rho} + \mathbf{g} + \rho\beta \frac{dK}{d\rho} \mathbf{u}\right) - C_1\beta \frac{dK}{d\rho} \mathbf{u}\right] dZ_\rho\right\} \tag{2.5}$$

Relations (2.4) and (2.5) enable us to express the spectral measures of all random processes under investigation here in terms of the spectral measure of the process ρ'. We see that these relations also fully define the Fourier transforms of various correlation functions which are of interest in our theory, provided that the spectral density of the random process ρ' is known.

3. Dynamics of the concentration fluctuations in a disperse system. Let us now consider the dynamics of variation of the random field $\rho'(\mathbf{r}, t)$.

This variation is obviously caused by chaotic pulsations of the particles taking part in the quasi-turbulent motion. In cases where the mass flux $\mathbf{J}(\mathbf{r}, t)$ of the particles caused by such motions is relatively small and where the characteristic time T_J associated with its variation greatly exceeds the internal quasi-turbulence time scale τ, the variation of $\rho(\mathbf{r}, t)$ can be described, as we know, with aid of the standard diffusion equation (*). The first of these assumptions generally represents the necessary condition of existence of some differential equation for $\rho'(\mathbf{r}, t)$, which becomes the well-known diffusion equation when the second assumption is satisfied (e.g. see [11]). In the argument that follows we consider the assumption of a small flux $\mathbf{J}(\mathbf{r}, t)$ adequate, but surrender the assumption that it varies slowly. This is necessary, since some of the random processes introduced above are generally absent [10] when the standard diffusion equation is used.

We shall use the method developed in [11] to allow for the subsequent term of expansion in τ / T_J in the diffusion equation. When the distribution function $f(\mathbf{w}; \mathbf{r}, t)$ exhibits a weak angular dependence, i.e. when the flux J is (as we assume) small, it can be written as

$$f(\mathbf{w}; \mathbf{r}, t) \approx \frac{1}{4\pi m} c^*(w; \mathbf{r}, t) + \frac{3}{4\pi m w^2} \mathbf{w}\mathbf{J}^*(w; \mathbf{r}, t) \tag{3.1}$$

where $c^* w^2 dw$ and $\mathbf{J}^* w^2 dw$ are the mass concentrations and the particle flux with its momentum lying in the range $(m\mathbf{w}, m(\mathbf{w} + d\mathbf{w}))$.

We shall carry out our analysis in a coordinate system attached to the mean motion of particles, neglecting the space and time dependences of the dynamic variables. Then, provided we disregard the "scattering" of the particles due to interactions with each other and to the fluctuations of the supporting stream, the following continuity equation [11] holds for $f(\mathbf{w}; \mathbf{r}, t)$:

$$\frac{\partial}{\partial t} f(\mathbf{w}; \mathbf{r}, t) = -\mathbf{w} \frac{\partial}{\partial \mathbf{r}} f(\mathbf{w}; \mathbf{r}, t) \tag{3.2}$$

To allow for such a scattering we must introduce the total effective scattering cross section per unit volume $n_t Q$. For an ordinary gas n_t represents the concentration of the molecular scattering centers, Q is the effective cross section of momentum transfer per scattering event, and the product $(n_t Q)^{-1}$ represents the mean free path of a particle between successive scattering events. The standard procedure now yields

$$\frac{\partial c^*}{\partial t} + \frac{3}{w^2} \mathbf{w} \frac{\partial \mathbf{J}^*}{\partial t} \approx -\mathbf{w} \frac{\partial c^*}{\partial \mathbf{r}} - \frac{3}{w^2} \mathbf{w} \frac{\partial (\mathbf{w}\mathbf{J}^*)}{\partial \mathbf{r}} - \frac{3}{w} n_t Q \mathbf{w}\mathbf{J}^* \tag{3.3}$$

This equation differs from the analogous equation in [11] only by the absence of terms containing the effective absorption cross section and the particle source function. It contains two types of terms: terms invariant under changes of the direction of $\mathbf{w}$ and terms which change sign. This implies that (3.3) is equivalent to the following two equations:

$$\frac{\partial c^*}{\partial t} \approx -\frac{3}{w^2} \mathbf{w} \frac{\partial (\mathbf{w}\mathbf{J}^*)}{\partial \mathbf{r}}, \qquad \frac{1}{w n_t Q} \frac{\partial \mathbf{J}^*}{\partial t} + \mathbf{J}^* \approx -\frac{w}{3 n_t Q} \frac{\partial c^*}{\partial \mathbf{r}} \tag{3.4}$$

Neglecting the first term in the second equation, we readily obtain the standard diffusion equation.

Equations (3.3) yield the following equation for the unknown c^*:

$$\frac{1}{w n_t Q} \frac{\partial^2 c^*}{\partial t^2} + \frac{\partial c^*}{\partial t} \approx \frac{1}{w n_t Q} \frac{\partial}{\partial \mathbf{r}} \left(\langle \mathbf{w} * \mathbf{w} \rangle \frac{\partial c^*}{\partial \mathbf{r}} \right) \tag{3.5}$$

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*) Such a method of describing $\rho'(t, \mathbf{r})$ was used earlier in [10], although only the diffusion resulting from the small-scale component of the pulsations of the suspended particles was taken into account.

Let us now introduce the quantities c and $\mathbf{J}$ (whose meaning is obvious) and the average quantities $c(\mathbf{r}, t) = \int w^2 c^*(w; \mathbf{r}, t)\, dw, \quad \mathbf{J}(\mathbf{r}, t) = \int w^2 \mathbf{J}^*(w; \mathbf{r}, t)\, dw$

$$\mathbf{D} = \| D_{ij} \|, \quad D_{ij} = \frac{1}{n_t Q} \frac{\langle w_i w_j \rangle}{w^*}, \quad w^* = \langle \mathbf{ww} \rangle^{1/2} \tag{3.6}$$

We can now show as in [11] that (3.5) is equivalent to the following generalized diffusion equation: (*)

$$\frac{\partial c}{\partial t} = \left[\frac{\partial}{\partial \mathbf{r}} \left(\mathbf{D} \frac{\partial}{\partial \mathbf{r}} \right) - \frac{\operatorname{tr} \mathbf{D}}{w^{*2}} \frac{\partial^2}{\partial t^2} \right] c, \quad \operatorname{tr} \mathbf{D} = D_{ii} \tag{3.7}$$

We can arrive at the same result by a fundamentally different method, namely by considering directly the problem of random motion of a particle in three-dimensional space under the assumption that the displacement velocity of the particle is finite. A detailed analysis of the one-dimensional case of this problem is carried out in [12].

From (3.7) we can obtain a similar equation for the volume concentration ρ of the particles, or for its perturbation ρ'. The solutions of the last equation describe the regular degeneration of this perturbation. The random appearance of such a perturbation of concentration caused due to fluctuations can be described by introducing a certain source function, which is Markovian with respect to time, into the right side of this equation. Such a function was already introduced in [1–3]. The resulting equation for the spectral measure of the process ρ' discussed in Sect. 2 yields the following relation:

$$dZ_\rho = dZ \left[i\omega + \left(\mathbf{Dkk} - \omega^2 \frac{D_{ii}}{w^{*2}} \right) \right]^{-1} \tag{3.8}$$

which closes the system of spectral equations obtained in Sect. 2. The spectral measure dZ appearing in (3.8) shares with the quantity Φ defined by the relation $\langle dZ^* dZ \rangle = \Phi d\omega d\mathbf{k}$ the property of depending on the wave vector $\mathbf{k}$ but not on the frequency ω. Relation (3.8) yields the following expression for the spectral density of the process ρ':

$$\Psi_{\rho,\rho}(\omega, \mathbf{k}) = \Phi_{\rho,\rho}(\mathbf{k}) \left[\omega^2 + \left(\mathbf{Dkk} - \omega^2 \frac{D_{ii}}{w^{*2}} \right)^2 \right]^{-1} \times$$

$$\times \left\{ \int \left[\omega^2 + \left(\mathbf{Dkk} - \omega^2 \frac{D_{ii}}{w^{*2}} \right)^2 \right]^{-1} d\omega \right\}^{-1}, \quad \Psi_{\rho,\rho}\, d\omega\, d\mathbf{k} = \langle dZ_\rho^*\, dZ_\rho \rangle \tag{3.9}$$

in which the partial spectral density $\Phi_{\rho,\rho}(\mathbf{k})$ defining the simultaneous correlation functions of the process ρ' is assumed to be known. The latter quantity can be represented in one of the forms discussed in [1–3, 10].

Relation (3.9) together with Eqs. (2.4) and (2.5) makes it possible to compute all the spectral densities which are of interest. These in turn yield (by standard methods) the corresponding correlation functions. Relations thus obtained contain quantities w^* and

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*) This corresponds to the case where the probability of the particle velocity change $\mathbf{w}_1 \to \mathbf{w}_2$ during a single scattering event depends only on the angle between the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$. The quantity $n_t Q$ is then a scalar. In the general case when the scattering is largely anisotropic in the sense that the above probability depends not only on the angle between $\mathbf{w}_1$ and $\mathbf{w}_2$ but also on their directions, we can introduce the tensor $\mathbf{n}_t\mathbf{Q}$ and the effective free-path length tensor $\boldsymbol{\lambda} = (\mathbf{n}_t\mathbf{Q})^{-1}$. In this case the formula (3.6), for example, is replaced by the expression $D_{ij} = w^{*-1} \lambda_{ik} \langle w_k w_j \rangle$. However, the tensor character of λ is not essential in our case, since formula (3.6) is not used in its explicit form below.

D_{ij} not known a priori. In computing them it is convenient to orient the coordinate axes in such a way, that the tensor $\langle w_i' w_j' \rangle$ is diagonal. We can see from (3.6) that the diffusion tensor $\mathbf{D}$ is also diagonal in such a coordinate system. Using (2.4), (2.5) and (3.9) we can readily find the quantity $\Psi_{wi,wi}$, representing the first invariant of the tensor spectral density of the process $\mathbf{w}'$. This quantity depends on w^* and on the eigenvalues D_i of the tensor $\mathbf{D}$ as on parameters. We have the self-evident equation

$$\int \Psi_{wi,wi}\, d\omega\, d\mathbf{k} = w^{*2}$$

defining w^* as a function of D_i, of the dynamic variables, and of the physical phase parameters. In an entirely analogous manner we can express D_{ij} as integrals of the corresponding components of $\Psi_{wi, wj}$ to obtain equations for D_i. Formulas (3.6) are very inconvenient for computing D_{ij}, since they contain $n_t Q$, which is not known. It is therefore expedient to obtain the relations for D_i by means of relations analogous to those for the turbulent diffusion coefficients. In this way we obtain

$$\int_0^\infty d\tau \int e^{i\omega\tau} \Psi_{wi,wj}\, d\omega\, d\mathbf{k} = D_{ij} = D_i, \quad i = j$$

The results obtained in this section enable us to complete our determination of the terms appearing in dynamic equations (1.4) and (1.7) as the result of quasi-turbulence. We see that these equations constitute an approximation corresponding to the Euler approximation in classical hydromechanics. Indeed, when analyzing the random processes in Sect. 2, we omitted from all the stochastic equations terms of the order of T / T_0 or L / L_0, where L denotes the external quasi-turbulence space scale, and where T_0 and L_0 are the characteristic dimensions of the average motion of a disperse system. The next approximation (analogous to the Navier-Stokes approximation in the hydrodynamics of a homogeneous fluid) could consist in admitting such terms directly into the above equations. An alternative method consists in the purely phenomenological addition of terms representing the stresses due to the quasi-turbulent viscosity [1, 2] to the equations of conservation of momentum of the average motion of phases. For example, in the momentum equation for the dispersed phase these terms can be written as

$$\frac{\partial \boldsymbol{\tau}^{(p)}}{d\mathbf{r}}, \quad \tau_{ij}^{(p)} = \eta_{ik}^{(p)} \frac{\partial w_j}{\partial x_k} + \eta_{jk}^{(p)} \frac{\partial w_i}{\partial x_k}, \quad \boldsymbol{\eta}^{(p)} = \rho d_2 \mathbf{D}^{(p)}, \quad \mathbf{D}^{(p)} = \mathbf{D}$$

We also note that expressions (1.4) for the quasi-turbulent pressure of the dispersed phase and for the components of the tensor $\boldsymbol{\eta}^{(p)}$ of the effective quasi-turbulent viscosity of this phase do not allow for instantaneous momentum transfer within the material of the suspended solid particles. Inclusion of the latter requires that $\mathbf{P}^{(p)}$ and $\boldsymbol{\eta}^{(p)}$ obtained here be multiplied by a definite function of the average volume concentration of the disperse system [3].

4. Equations for the pulsation energy of the phases. So far we have discussed only the "equilibrium" states of the quasi-turbulent motion, in the sense that we have assumed the dynamic variables to be independent of the time and coordinates and that the motions themselves were fully steadystate. In reality, when the dynamic parameters are variable (although we shall still assume that their derivatives are small) and when the stream is bounded (by walls, etc.), the resulting quasi-turbulence can deviate appreciably from the steady state. We distinguish two different relaxation processes: (1) establishment of equilibrium at the level of individiual particles and of the fluid contained within their specific volumes, so that the local state of the system can be defined in terms of certain given mean quasi-turbulence characteristics; (2) the relaxation of these mean values to their equilibrium values (obtained above) whose variations follow those of the dynamic variables. Clearly, the scale of the first relaxation process coincides with the internal scale τ and that of the second process with the external quasi-turbulence time scale T. In the asymptotic case being considered here $t \gg \tau$. This obliges us to neglect the first process altogether and to consider only the states which are "relaxed" in the sense that they can be referred to as states "with local equilibrium".

We note that the first relaxation process is analogous to the establishment of local equilibrium in ordinary thermodynamic systems (e.g. to establishment of a state of molecular chaos in the kinetic theory of gases) which we can generally describe in terms of the mean characteristics of the molecular motion, e.g. temperature of the system. The other relaxation process is analogous to the process of smoothing out these mean values (e.g. of thermal conductivity). Moreover, ignoring the first process in gas-solid mixtures is exactly equivalent to postulating the presence of local thermodynamic equilibrium in the hydromechanics of a single-phase fluid.

Obviously the quantity $\langle \rho'^2 \rangle$ is, by definition, independent of the level of development of the quasi-turbulent motions. We assume that the statistical characteristics of the pressure pulsations p' in the state of local equilibrium are identical with those in the corresponding equilibrium state. This means that the relaxation of p' is due largely to the first relaxation process. The level of development of quasi-turbulence in the state of local equilibrium can be described with the aid of the quantities

$$\vartheta_i = \langle v_i'^2 \rangle, \quad \theta_i = \langle w_i'^2 \rangle$$

representing the energy of the pulsations of the fluid and of the dispersed phase in various directions. We consider these quantities as some unknown functions of t and $\mathbf{r}$, which become ϑ_i° and θ_i° in the equilibrium state. Obviously ϑ_i° and θ_i° are functions of the dynamic variables, i.e. they depend on t and $\mathbf{r}$ implicitly only.

Our task is to obtain specific equations for ϑ_i and θ_i. This generally requires the ability to describe in detail the interaction processes of duration $\sim \tau$ in the system. If these details are not available, then we must either make some additional assumptions about the character of this interaction, or else make some assumptions about the first relaxation process. A very simple approximate model of this type can be based on the assumption that the above relaxation process leads to a smoothing of the velocities of the quasi-turbulent motion in various directions such that ϑ_i or θ_i behave exactly as ϑ_i° and θ_i° and that the phase characteristics of the random processes coincide with those in the corresponding equilibrium state. We then have the following relations:

$$\vartheta_i = \vartheta (\vartheta_i^\circ / \vartheta^\circ), \quad \theta_i = \theta (\theta_i^\circ / \theta^\circ), \quad \vartheta = \Sigma_i \vartheta_i, \quad \theta = \Sigma_i \theta_i$$

$$\langle v_i' v_j' \rangle = \frac{\vartheta}{\vartheta^\circ} \langle v_i' v_j' \rangle^\circ, \quad \langle \rho' v_i' \rangle = \left(\frac{\vartheta}{\vartheta^\circ}\right)^{1/2} \langle \rho' v_i' \rangle^\circ \tag{4.1}$$

$$\langle w_i' w_j' \rangle = \frac{\theta}{\theta^\circ} \langle w_i' w_j' \rangle^\circ, \quad \langle \rho' w_i' \rangle = \left(\frac{\theta}{\theta^\circ}\right)^{1/2} \langle \rho' w_i' \rangle^\circ$$

$$\left\langle v_i' \frac{\partial \pi'}{\partial x_j} \right\rangle = \left(\frac{\vartheta}{\vartheta^\circ}\right)^{1/2} \left\langle v_i' \frac{\partial \pi'}{\partial x_j} \right\rangle^\circ, \quad \left\langle w_i' \frac{\partial \pi'}{\partial x_j} \right\rangle = \left(\frac{\theta}{\theta^\circ}\right)^{1/2} \left\langle w_i' \frac{\partial \pi'}{\partial x_j} \right\rangle^\circ$$

Here and below the superscript ° denotes the quantities in the equilibrium state.

For simplicity we shall consider only the equations for ϑ and θ , neglecting the force $\mathbf{F}_B'$ in (2.2) (*). From the second equation of (2.2) we obtain

$$(1+\varkappa\xi)\, v_i' \frac{\partial w_i'}{\partial t} = \frac{\varkappa\xi}{2}\frac{\partial v}{\partial t} + \mathbf{v}'\mathbf{W}', \quad \mathbf{W}' = -\frac{\partial \pi'}{\partial \mathbf{r}} + \varkappa\beta\left(K\mathbf{u}' + \frac{dK}{d\rho}\,\mathbf{u}\rho'\right) \quad (4.2)$$

Multiplying the second and third equations of (2.2) by w_i' and v_i' resepectively, applying (4.2), and averaging, we obtain

$$\frac{\varkappa}{2}(1-\rho+\rho\xi)\left(\frac{\partial}{\partial t} + \mathbf{u}\frac{\partial}{\partial \mathbf{r}}\right)\vartheta - \frac{\varkappa\rho\xi}{2}\frac{\varkappa\xi}{1+\varkappa\xi}\frac{\partial\vartheta}{\partial t} - \frac{\varkappa\nu}{2}\Delta\vartheta = \left\langle \mathbf{v}'\left(\mathbf{V}' + \frac{\varkappa\rho\xi}{1+\varkappa\xi}\mathbf{W}'\right)\right\rangle \quad (4.3)$$

$$\frac{1+\varkappa\xi}{2}\frac{\partial\theta}{\partial t} + \frac{\varkappa\xi}{2}\frac{\varkappa\xi}{1+\varkappa\xi}\frac{\partial\vartheta}{\partial t} - \varkappa\xi\frac{\partial}{\partial t}\langle\mathbf{v}'\mathbf{w}'\rangle = \left\langle\left(\mathbf{w}' - \frac{\varkappa\xi}{1+\varkappa\xi}\mathbf{v}'\right)\mathbf{W}'\right\rangle$$

$$\mathbf{V}' = -(1-\rho)\frac{\partial\pi'}{\partial\mathbf{r}} - \varkappa\rho\beta\left(K\mathbf{u}' + \frac{dK}{d\rho}\mathbf{u}\rho'\right) - \varkappa g\rho'$$

For example, let us consider $\langle \mathbf{v}'\mathbf{W}'\rangle$ and $\langle \mathbf{w}'\mathbf{W}'\rangle$. Using (4.1) we obtain

$$\langle\mathbf{v}'\mathbf{W}'\rangle = \sqrt{\vartheta}\left[\varkappa\beta K\left(\sqrt{\vartheta} - \frac{\langle\mathbf{v}'\mathbf{w}'\rangle^\circ}{\sqrt{\vartheta^\circ\theta^\circ}}\sqrt{\theta}\right) + \frac{\varkappa\beta\mathbf{u}}{\sqrt{\vartheta^\circ}}\frac{dK}{d\rho}\langle\rho'\mathbf{v}'\rangle^\circ - \frac{1}{\sqrt{\vartheta^\circ}}\left\langle\mathbf{v}'\frac{\partial\pi'}{\partial\mathbf{r}}\right\rangle^\circ\right]$$

$$\langle\mathbf{w}'\mathbf{W}'\rangle = \sqrt{\theta}\left[\varkappa\beta K\left(\frac{\langle\mathbf{v}'\mathbf{w}'\rangle^\circ}{\sqrt{\vartheta^\circ\theta^\circ}}\sqrt{\vartheta} - \sqrt{\theta}\right) + \frac{\varkappa\beta\mathbf{u}}{\sqrt{\theta^\circ}}\frac{dK}{d\rho}\langle\rho'\mathbf{w}'\rangle^\circ - \frac{1}{\sqrt{\theta^\circ}}\left\langle\mathbf{w}'\frac{\partial\pi'}{\partial\mathbf{r}}\right\rangle^\circ\right] \quad (4.4)$$

But Eqs. (2.2) or (4.3) written out for the equilibrium state when the dynamic variables have constant values, yield

$$-\left\langle\mathbf{v}'\frac{\partial\pi'}{\partial\mathbf{r}}\right\rangle^\circ + \varkappa\beta\mathbf{u}\frac{dK}{d\rho}\langle\rho'\mathbf{v}'\rangle^\circ = -\varkappa\beta K\sqrt{\vartheta^\circ}\left(\sqrt{\vartheta^\circ} - \frac{\langle\mathbf{v}'\mathbf{w}'\rangle^\circ}{\sqrt{\vartheta^\circ\theta^\circ}}\sqrt{\theta^\circ}\right)$$

$$-\left\langle\mathbf{w}'\frac{\partial\pi'}{\partial\mathbf{r}}\right\rangle^\circ + \varkappa\beta\mathbf{u}\frac{dK}{d\rho}\langle\rho'\mathbf{w}'\rangle^\circ = -\varkappa\beta K\sqrt{\theta^\circ}\left(\frac{\langle\mathbf{v}'\mathbf{w}'\rangle^\circ}{\sqrt{\vartheta^\circ\theta^\circ}}\sqrt{\vartheta^\circ} - \sqrt{\theta^\circ}\right) \quad (4.5)$$

which enable us to transform (4.4). The same operation can readily be carried out for the quantities $\langle\mathbf{v}'\mathbf{V}'\rangle$ and $\langle\mathbf{w}'\mathbf{V}'\rangle$.

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*) If the force $\mathbf{F}_B'$ must be retained (e.g. for large particles), then the following time correlation functions $\langle\mathbf{v}'(t+\tau)\mathbf{v}'(t)\rangle$ $\langle\mathbf{v}'(t+\tau)\mathbf{w}'(t)\rangle$, $\langle\mathbf{w}'(t+\tau)\mathbf{w}'(t)\rangle$ occur in the equations for ϑ and θ In this case we must either construct the equations for these correlation functions by means of the familiar rules, or we must assume as in (4.1), that

$$\langle\mathbf{v}'(t+\tau)\mathbf{v}'(t)\rangle = (\vartheta/\vartheta^\circ)\,\langle\mathbf{v}'(t+\tau)\mathbf{v}'(t)\rangle^\circ$$

$$\langle\mathbf{v}'(t+\tau)\mathbf{w}'(t)\rangle = (\vartheta\theta/\vartheta^\circ\theta)^{1/2}\,\langle\mathbf{v}'(t+\tau)\mathbf{w}'(t)\rangle^\circ$$

$$\langle\mathbf{w}'(t+\tau)\mathbf{w}'(t)\rangle = (\theta/\theta^\circ)\,\langle\mathbf{w}'(t+\tau)\mathbf{w}'(t)\rangle^\circ$$

As a result, we obtain the following equations from (4.3):

$$\frac{\varkappa(1-\rho)}{2}\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\vartheta-\frac{\varkappa\xi\rho}{2}\frac{\varkappa\xi}{1+\varkappa\xi}\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\vartheta-\frac{\varkappa\nu}{2}\Delta\vartheta=R_f \quad (4.6)$$

$$\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\left[\frac{1+\varkappa\xi}{2}\theta+\frac{\varkappa\xi}{2}\frac{\varkappa\xi}{1+\varkappa\xi}\vartheta-\varkappa\xi\left(\frac{\langle\mathbf{v}'\mathbf{w}'\rangle^\circ}{\sqrt{\vartheta^\circ\theta^\circ}}\sqrt{\vartheta\theta}\right)\right]=R_p$$

$$R_f=\frac{\varkappa\rho}{1+\varkappa\xi}\beta K\sqrt{\bar\vartheta}\left[\sqrt{\bar\vartheta^\circ}-\sqrt{\bar\vartheta}-\frac{\langle\mathbf{v}'\mathbf{w}'\rangle^\circ}{\sqrt{\vartheta^\circ\theta^\circ}}\left(\sqrt{\bar\theta^\circ}-\sqrt{\bar\theta}\right)\right]$$

$$R_p=\varkappa\beta K\left\{\sqrt{\bar\theta}\left[\sqrt{\bar\theta^\circ}-\sqrt{\bar\theta}-\frac{\langle\mathbf{v}'\mathbf{w}'\rangle^\circ}{\sqrt{\vartheta^\circ\theta^\circ}}\left(\sqrt{\bar\vartheta^\circ}-\sqrt{\bar\vartheta}\right)\right]+\right.$$

$$\left.+\frac{\varkappa\xi}{1+\varkappa\xi}\sqrt{\bar\vartheta}\left[\sqrt{\bar\vartheta^\circ}-\sqrt{\bar\vartheta}-\frac{\langle\mathbf{v}'\mathbf{w}'\rangle^\circ}{\sqrt{\vartheta^\circ\theta^\circ}}\left(\sqrt{\bar\theta^\circ}-\sqrt{\bar\theta}\right)\right]\right\}$$

which for suspensions of particles in gases reduce to a single equation for θ ($\varkappa\to 0$, $\varkappa\beta\neq 0$)

$$\left(\frac{\partial}{\partial t}+\mathbf{w}\frac{\partial}{\partial \mathbf{r}}\right)\sqrt{\bar\theta}\approx\varkappa\beta K\left[1-\left(\frac{\langle\mathbf{v}'\mathbf{w}'\rangle^\circ}{\sqrt{\vartheta^\circ\theta^\circ}}\right)^2\right]\left(\sqrt{\bar\theta^\circ}-\sqrt{\bar\theta}\right) \quad (4.7)$$

Unlike (4.3), Eqs. (4.6) and (4.7) are written out in the laboratory coordinate system where $\mathbf{w}\neq 0$. Together with dynamic equations (1.4) and (1.7) they fully define the state of the system.

Our energy equations are also in the Euler approximation. To obtain the next approximation we must take into account: (1) the increase in the quasi-turbulent energy due to dissipation of the energy of the average motion through pulsations ; (2) the transfer of pulsation energy by the pulsations themselves. These factors can be allowed for phenomenologically by introducing the effective quasi-turbulent viscosity tensors and the effective coefficients of quasi-turbulent phase energy transfer. For example, R_p in the second equation of (4.6) is here replaced by

$$R_p+\frac{\partial}{\partial \mathbf{r}}\left(\boldsymbol{\lambda}^{(p)}\frac{\partial\theta}{\partial \mathbf{r}}\right)+\mathbf{w}\frac{\partial\boldsymbol{\tau}^{(p)}}{\partial \mathbf{r}}$$

Using the dynamic equations together with the quasi-turbulent energy transfer equations, we can readily obtain the equations for the energy of the mean phase motion of a disperse system.

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ANALYSIS OF THE THREE-DIMENSIONAL STATES OF STRESS AND STRAIN OF CIRCULAR CYLINDRICAL SHELLS. CONSTRUCTION OF REFINED APPLIED THEORIES

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The nonaxisymmetric problem of elasticity theory for circular cylindrical shells loaded along the endface surface Γ_2 is considered. By using the method of trigonometric series expansions, homogeneous solutions of closed $(\Gamma_2 : z = \pm l)$ and open $(\Gamma_2 : \varphi = + \varphi_0)$ shells are studied as their thickness decreases.

It is proved that the state of stress of a closed shell includes four parts: (1) an elementary state of stress penetrating into the shell without attenuation, (2) a slowly attenuated principal state of stress, (3) a rapidly attenuating state of stress (edge effect of shells), (4) a boundary layer type of state of stress.

In the case of an open cylindrical shell subjected to a periodic loading with period l_0, there are states of stress of types (1), (3) and (4). The rate of attenuation of the edge effects hence depend essentially on the number of the term of the trigonometric series as well as on the quantity l_0. In both cases asymptotic expansions are presented of the components of the states of stress and strain.

On the basis of the exact solution of the three-dimensional problem, a refined applied theory is given for a circular cylindrical shell, which is intended to reduce the stress from the endface surface Γ_2. Applied theories reducing the stresses from the cylindrical portions of the shell boundary were considered earlier in [1].

1. Construction of homogeneous solutions. Let us consider the arbitrary strain of an elastic isotropic shell bounded by coaxial circular cylinders Γ_1 of radii R_1 and R_2 $(R_1 < R_2)$ and an endface surface Γ_2. Let us assume that the stress resultants applied to the boundary Γ_2 form a system statically equivalent to zero, and the boundary Γ_1 is stress-free. As the initial relationships let us take expressions for the